Inversione della mappa di Fueter-Sce

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The Fueter-Sce Mapping theorem

Notations

Let $\mathbb{R}_n$ be the real Clifford algebra over n imaginary units $e_1, \dots, e_n$ satisfying the relations

$$e_i e_j + e_j e_i = 0, \ \ i \neq j \qquad e_i^2 = -1.$$

An element in the Clifford algebra will be denoted by

$$\sum_A e_A x_A$$

where

$$A = \{i_1 \dots i_r\} \in \mathcal{P}\{1, 2, \dots, n\}, \quad i_1 < \dots < i_r$$

is a multi-index and $e_A = e_{i_1} e_{i_2} \dots e_{i_r}$, $e_\emptyset = 1$.

The Fueter-Sce Mapping theorem

An element $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ will be identified with the element

$$x = x_0 + \underline{x} = x_0 + \sum_{j=1}^{n} x_j e_j \in \mathbb{R}_n$$

called, in short, paravector. The norm of $x \in \mathbb{R}^{n+1}$ is defined as

$$|x|^2 = x_0^2 + x_1^2 + \ldots + x_n^2.$$

The real part x_0 of x is also denoted by $\mathrm{Re}[x]$; $\underline{x}$ is the 1-vector part of x; the conjugate of x is defined by $\bar{x} = x_0 - \underline{x} = x_0 - \sum_{j=1}^{n} x_j e_j$.

The Fueter-Sce Mapping theorem

The sphere $\mathbb{S}^{n-1}$

$\mathbb{S}^{n-1} = \{\underline{x} = e_1 x_1 + \ldots + e_n x_n \mid x_1^2 + \ldots + x_n^2 = 1\}$ $\underline{\omega} \in \mathbb{S}^{n-1}$, $\underline{\omega}^2 = -1$

The complex plane $\mathbb{C}_{\underline{\omega}}$

The vector space $\mathbb{R} + \underline{\omega}\mathbb{R}$ passing through 1 and $\underline{\omega} \in \mathbb{S}^{n-1}$ will be denoted by $\mathbb{C}_{\underline{\omega}}$, while an element belonging to $\mathbb{C}_{\underline{\omega}}$ will be denoted by $u + \underline{\omega}v$, for u, $v \in \mathbb{R}$. $\mathbb{C}_{\underline{\omega}}$ can be identified with a complex plane.

The Fueter-Sce Mapping theorem

The sphere $\mathbb{S}^{n-1}$

$\mathbb{S}^{n-1} = \{\underline{x} = e_1x_1 + \ldots + e_nx_n \mid x_1^2 + \ldots + x_n^2 = 1\}$ $\underline{\omega} \in \mathbb{S}^{n-1}$, $\underline{\omega}^2 = -1$

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The vector space $\mathbb{R} + \underline{\omega}\mathbb{R}$ passing through 1 and $\underline{\omega} \in \mathbb{S}^{n-1}$ will be denoted by $\mathbb{C}_{\underline{\omega}}$, while an element belonging to $\mathbb{C}_{\underline{\omega}}$ will be denoted by $u + \underline{\omega}v$, for u, $v \in \mathbb{R}$. $\mathbb{C}_{\underline{\omega}}$ can be identified with a complex plane.

The Fueter-Sce Mapping theorem

Proposition

Let f be an holomorphic function in an open set of the upper half complex plane

$$f(x + \iota y) = u(x, y) + \iota v(x, y)$$

$$q = x_0 + ix_1 + jx_2 + kx_3 := x_0 + \underline{q}$$

then

$$\Delta_4 (u(x_0, |\underline{q}|) + \frac{\underline{q}}{|\underline{q}|} v(x_0, |\underline{q}|))$$

is Fueter regular, while when $x_0 + \underline{x} \in \mathbb{R}^{n+1}$

$$\Delta_{n+1}^{\frac{n-1}{2}} (u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|))$$

is in the kernel of Dirac operator $\partial_x = \partial_{x_0} + \sum_i e_i \partial_{x_i}$.

The Fueter-Sce Mapping theorem

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The inverse Fueter mapping theorem, I

Intrinsic functions

Let $f(u+\iota v)=\alpha(u,v)+\iota\beta(u,v)$ be a function defined for $z=u+\iota v\in\mathcal{U}\subseteq\mathbb{C}$, $\mathcal{U}$ symmetric with respect to the real axis. Assume $\alpha(u,-v)=\alpha(u,v)$, $\beta(u,-v)=-\beta(u,v)$, (α,β) satisfying the Cauchy-Riemann equations.

Definition

Let $U=\{x=x_0+\underline{x}\in\mathbb{R}^{n+1}\mid(x_0,|\underline{x}|)\in\mathcal{U}\}$ and let $\mathcal{SM}(U)=\{f:U\subseteq\mathbb{R}^{n+1}\to\mathbb{R}_n,\ f(x)=f(x_0+\underline{\omega}|\underline{x}|)=\alpha(x_0,|\underline{x}|)+\underline{\omega}\beta(x_0,|\underline{x}|)\ \alpha,\ \beta\ \mathbb{R}_n$-valued and with the above properties $\}$.

The inverse Fueter mapping theorem, I

Definition (Axially monogenic function)

Let U be an axially symmetric open set in $\mathbb{R}^{n+1}$, and let $x = x_0 + \underline{x} = x_0 + r\underline{\omega} \in U$. We say that $\tilde{f}$ is an axially monogenic function if there exist two functions $A = A(x_0, r)$ and $B = B(x_0, r)$, independent of $\underline{\omega} \in \mathbb{S}^{n-1}$ and with values in $\mathbb{R}_n$, such that

$$\tilde{f}(x) = A(x_0, r) + \underline{\omega} B(x_0, r),$$

and $\tilde{f}$ is a monogenic function, that is it is in the kernel of the Dirac operator. We denote by $\mathcal{AM}(U)$ the set of left axially monogenic functions on the open set U.

The inverse Fueter mapping theorem, I

Theorem

Let U be an axially symmetric open set in $\mathbb{R}^{n+1}$. Then the functions $A = A(x_0, r)$ and $B = B(x_0, r)$ satisfy the Vekua's system, i.e.

$$\begin{cases} \partial_{x_0} A(x_0, r) - \partial_r B(x_0, r) = \frac{n-1}{r} B(x_0, r), \\ \partial_{x_0} B(x_0, r) + \partial_r A(x_0, r) = 0. \end{cases}$$

The inverse Fueter mapping theorem, I

Problem (The inverse Fueter mapping)

Let n be an odd number and let U be a suitable open set in $\mathbb{R}^{n+1}$. Given an axially monogenic function $\tilde{f}$, find a function $f \in \mathcal{SM}(U)$ such that

$$\tilde{f}(x) = \Delta^{\frac{n-1}{2}} f(x).$$

Find an integral representation of the map

$$\mathcal{AM}(U) \to \mathcal{SM}(U), \qquad \tilde{f} \mapsto f.$$

The inverse Fueter mapping theorem, I

Definition

Let n be an odd number and let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set. Suppose that $f \in \mathcal{SM}(U)$. We say that a function f is a Fueter primitive of $\tilde{f} \in \mathcal{AM}(U)$ if

$$\Delta^{\frac{n-1}{2}} f(x) = \tilde{f}(x)$$

on U.

The inverse Fueter mapping theorem, I

Definition (The functions $\mathcal{N}_n^+(x)$ and $\mathcal{N}_n^-(x)$)

Let $\mathcal{G}(x-\underline{y})$ be the monogenic Cauchy kernel with $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$ and we assume $\underline{y} = \underline{\omega} \in \mathbb{R}^n$, $\underline{\omega} \in \mathbb{S}^{n-1}$. We define

$$\mathcal{N}_n^+(x) = \int_{\mathbb{S}^{n-1}} \mathcal{G}(x-\underline{\omega})\, dS(\underline{\omega}), \qquad \mathcal{N}_n^-(x) = \int_{\mathbb{S}^{n-1}} \mathcal{G}(x-\underline{\omega})\,\underline{\omega}\, dS(\underline{\omega}).$$

where $dS(\underline{\omega})$ is the scalar element of surface area of $\mathbb{S}^{n-1}$.

The inverse Fueter mapping theorem, I

Theorem (The restrictions of $\mathcal{N}_n^+(x)$ and $\mathcal{N}_n^-(x)$ to $\underline{x} = 0$)

Let n be an odd number. Let $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$ be the functions defined above. Then their restrictions to $\underline{x} = 0$ are given by

$$\mathcal{N}_n^+(x)|_{\underline{x}=0} = \mathcal{C}_n \ \frac{x_0}{(x_0^2+1)^{(n+1)/2}}, \qquad \mathcal{N}_n^-(x)|_{\underline{x}=0} = -\mathcal{C}_n \ \frac{1}{(x_0^2+1)^{(n+1)/2}},$$

where

$$\mathcal{C}_n := \frac{1}{\sqrt{\pi}} \ \frac{\Gamma((n+1)/2)}{\Gamma(n/2)}. \tag{1}$$

The inverse Fueter mapping theorem, I

The structure of the Fueter primitives of $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$

Let n be an odd number and denote by $\mathcal{W}_n^+$ and $\mathcal{W}_n^-$ the Fueter primitives of $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$, respectively. Consider the functions:

$$\mathcal{W}_n^+(x_0) := \frac{\mathcal{C}_n}{\mathcal{K}_n}\ D^{-(n-1)} \frac{x_0}{(x_0^2+1)^{(n+1)/2}},$$

$$\mathcal{W}_n^-(x_0) := -\frac{\mathcal{C}_n}{\mathcal{K}_n}\ D^{-(n-1)} \frac{1}{(x_0^2+1)^{(n+1)/2}},$$

where the symbol $D^{-(n-1)}$ stands for the $(n-1)$ integrations with respect to x_0. Then replacing x_0 by x in $\mathcal{W}_n^+(x_0)$ and in $\mathcal{W}_n^-(x_0)$ we get $\mathcal{W}_n^+(x)$ and $\mathcal{W}_n^-(x)$, respectively.

The inverse Fueter mapping theorem, I

Corollary (Explicit Fueter's primitives of $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$ for $n = 3$)

$$\mathcal{W}_3^+(x) = \frac{1}{2\pi} \arctan x, \qquad \mathcal{W}_3^-(x) = -\frac{1}{2\pi}\, x \arctan x.$$

The inverse Fueter mapping theorem, I

Theorem (The inverse Fueter mapping theorem)

Let $\tilde{f}(x) = A(x_0, r) + \underline{\omega} B(x_0, r)$ be an axially monogenic function defined on an axially symmetric open set $U \subseteq \mathbb{R}^{n+1}$. Let Γ be the boundary of an open bounded subset $\mathcal{V}$ of the half plane $\mathbb{R} + \underline{\omega}\mathbb{R}^+$ and let $V = \{x = u + \underline{\omega} v,\ (u, v) \in \mathcal{V},\ \underline{\omega} \in \mathbb{S}^{n-1}\} \subset U$. Moreover suppose that Γ is a regular curve whose parametric equations $y_0 = y_0(s)$, $r = r(s)$. Then the function

$$f(x) = \int_\Gamma \mathcal{W}_n^-\Big(\frac{1}{r}(x - y_0)\Big) r^{n-2}(dy_0\, A(y_0, r) - dr\, B(y_0, r))$$

$$- \int_\Gamma \mathcal{W}_n^+\Big(\frac{1}{r}(x - y_0)\Big) r^{n-2}(dy_0 B(y_0, r) - drA(y_0, r)). \qquad (2)$$

is a Fueter's primitive of $\tilde{f}(x)$ on V, where $\mathcal{W}_n^+$ and $\mathcal{W}_n^-$ are Fueter primitives of $\mathcal{N}_n^+(x)$ and $\mathcal{N}_n^-(x)$, respectively.

The inverse Fueter mapping theorem, I

Theorem (The inverse Fueter mapping theorem for the quaternionic case)

Let $\tilde{f}(q) = A(q_0, r) + \underline{\omega} B(q_0, r)$ be an axially Fueter regular function defined on an axially symmetric domain $U \subseteq \mathbb{H}$. Let Γ be the boundary of an open bounded subset $\mathcal{V}$ of the half plane $\mathbb{R} + \underline{\omega}\mathbb{R}^+$ and let $V = \{x = u + \underline{\omega} v,\ (u, v) \in \mathcal{V},\ \underline{\omega} \in \mathbb{S}^{n-1}\} \subset U$. Moreover suppose that Γ is a regular curve . Then the function

$$f(q) = \int_\Gamma \mathcal{W}^- \Big(\frac{1}{r}(q - y_0)\Big)\, r\, (dy_0\, A(y_0, r) - dr\, B(y_0, r))$$

$$- \int_\Gamma \mathcal{W}^+ \Big(\frac{1}{r}(q - y_0)\Big)\, r\, (dy_0 B(y_0, r) - dr A(y_0, r)). \qquad (3)$$

is a Fueter primitive of $\tilde{f}(q)$ on V, where $\mathcal{W}^+(q) = \frac{1}{2\pi}$ arctan q and $\mathcal{W}^- = -\frac{1}{2\pi}\ q \arctan q$.

The inverse Fueter mapping theorem, II

Definition

A function $\tilde{f}_k(x)$ is said to be an axially monogenic function of degree k if it is of the form

$$\tilde{f}_k(x) = A_k(x_0, r, \underline{\omega}) + \underline{\omega} B_k(x_0, r, \underline{\omega})$$

where $A_k(x_0, r, \underline{\omega})$ and $B_k(x_0, r, \underline{\omega})$ satisfy the Vekua-type system:

$$\begin{cases} \partial_{x_0} A_k - \partial_r B_k = \frac{k+n-1}{r} B_k, \\ \partial_{x_0} B_k + \partial_r A_k = \frac{k}{r} A_k. \end{cases} \qquad (4)$$

The inverse Fueter mapping theorem, II

Definition

A left monogenic polynomial $\mathcal{P}_k$ in $\mathbb{R}^n$ is called inner spherical monogenic of degree k if it is homogeneous of degree k, that is $\mathcal{P}_k(\underline{x}/|\underline{x}|)|\underline{x}|^k$ and it satisfies $\partial_{\underline{x}}\mathcal{P}_k(\underline{x}) = 0$.

Let $A_k(x_0, r, \underline{\omega}) + \underline{\omega} B_k(x_0, r, \underline{\omega})$ be an axially monogenic function of degree k. For any (x_0, r) fixed the functions A_k and B_k are inner spherical monogenic of degree k in $\underline{\omega}$ and can be written as

$$A_k(x_0, r, \underline{\omega}) = A(x_0, r)\mathcal{P}_k(\underline{\omega}), \qquad B_k(x_0, r, \underline{\omega}) = B(x_0, r)\mathcal{P}_k(\underline{\omega}). \tag{5}$$

The inverse Fueter mapping theorem, II

Definition

A left monogenic polynomial $\mathcal{P}_k$ in $\mathbb{R}^n$ is called inner spherical monogenic of degree k if it is homogeneous of degree k, that is $\mathcal{P}_k(\underline{x}/|\underline{x}|)|\underline{x}|^k$ and it satisfies $\partial_{\underline{x}}\mathcal{P}_k(\underline{x}) = 0$.

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$$A_k(x_0, r, \underline{\omega}) = A(x_0, r)\mathcal{P}_k(\underline{\omega}), \qquad B_k(x_0, r, \underline{\omega}) = B(x_0, r)\mathcal{P}_k(\underline{\omega}). \quad (5)$$

The inverse Fueter mapping theorem, II

Theorem

Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set. Then every monogenic function $\tilde{f} : U \to \mathbb{R}_n$ can be written in the form $\tilde{f}(x) = \sum_{k=0}^{\infty} \tilde{f}_k(x)$ with

$$\tilde{f}_k(x) = \sum_{j=1}^{m_k} [A_{k,j}(x_0, r) + \underline{\omega} B_{k,j}(x_0, r)] \mathcal{P}_{k,j}(\underline{\omega}) \qquad (6)$$

where $\mathcal{P}_{k,j}$ form a basis for the space of spherical monogenics of degree k of dimension m_k and $A_{k,j}$, $B_{k,j}$ are suitable real valued functions.

The inverse Fueter mapping theorem, II

Problem

Find the inverse of the Fueter mapping theorem in the case of monogenic functions of type $(A_{k,j}(x_0, r) + \underline{\omega} B_{k,j}(x_0, r))\mathcal{P}_{k,j}(\underline{\omega})$ by providing their Fueter primitive.

$\mathcal{N}(U) = \{f : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n,\ f(x) = f(x_0 + I|\underline{x}|) = \alpha(x_0, |\underline{x}|) + \underline{\omega}\beta(x_0, |\underline{x}|) \mid \ \alpha(u, v) + \iota\beta(u, v)$ is a $\mathbb{C}$-valued holomorphic function in $u + \iota v \in \mathcal{U}\}$.

The inverse Fueter mapping theorem, II

Definition (Fueter's Primitive)

Let n be an odd number and let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric domain. Let $\tilde{f}(x)\mathcal{P}_k(\underline{x}) = (A(x_0, r) + \underline{\omega} B(x_0, r))\mathcal{P}_k(\underline{x})$ be an axially monogenic function of degree $k \in \mathbb{N}_0$. We say that a function $f(x)\mathcal{P}_k(\underline{x})$, $f \in \mathcal{N}(U)$ is a Fueter primitive of $\tilde{f}(x)\mathcal{P}_k(\underline{x})$ if

$$\Delta^{k+\frac{n-1}{2}}(f(x)\mathcal{P}_k(\underline{x})) = \tilde{f}(x)\mathcal{P}_k(\underline{x}) \quad on \quad U,$$

where Δ is the Laplace operator in dimension $n+1$.

The inverse Fueter mapping theorem, II

Definition (The functions $\mathcal{F}^+_{k,n}(x)$ and $\mathcal{F}^-_{k,n}(x)$)

Let $\mathcal{G}(x-\underline{y})$ be the monogenic Cauchy kernel with $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$ and for $\underline{y} = r\underline{\omega} \in \mathbb{R}^n$ we assume $r=1$ and $\underline{\omega} \in \mathbb{S}^{n-1}$. Let $\mathcal{P}_k(\underline{x})$ be an inner left spherical monogenic polynomial of degree $k \in \mathbb{N}_0$. We define

$$\mathcal{F}^+_{k,n}(x) = \int_{\mathbb{S}^{n-1}} \mathcal{G}(x-\underline{\omega})\, \mathcal{P}_k(\underline{\omega})\, dS(\underline{\omega}),$$

$$\mathcal{F}^-_{k,n}(x) = \int_{\mathbb{S}^{n-1}} \mathcal{G}(x-\underline{\omega})\, \underline{\omega}\; \mathcal{P}_k(\underline{\omega}) dS(\underline{\omega}),$$

where $dS(\underline{\omega})$ is the scalar element of surface area of $\mathbb{S}^{n-1}$.

The inverse Fueter mapping theorem, II

Theorem (Factorization property of $\mathcal{F}^{+}_{k,n}(x)$ and $\mathcal{F}^{-}_{k,n}(x)$)

Let n be an odd number. Let $\mathcal{P}_k(\underline{x})$ be an inner left spherical monogenic polynomial of degree $k \in \mathbb{N}_0$. Then there exists two functions $\mathcal{S}^{+}_{k,n}(x)$ and $\mathcal{S}^{-}_{k,n}(x)$ belonging to $\mathcal{N}(U)$, independent of $\mathcal{P}_k(\underline{x})$, such that
$\mathcal{F}^{+}_{k,n}(x) = \mathcal{S}^{+}_{k,n}(x)\mathcal{P}_k(\underline{x})$,
$\mathcal{F}^{-}_{k,n}(x) = \mathcal{S}^{-}_{k,n}(x)\mathcal{P}_k(\underline{x})$ *and*

$$\lim_{\underline{x}\to\underline{0}} \mathcal{S}^{+}_{k,n}(x) = \mathcal{C}_{k,n}\frac{x_0}{(x_0^2+1)^{k+(n+1)/2}},$$

$$\lim_{\underline{x}\to\underline{0}} \mathcal{S}^{-}_{k,n}(x) = -\mathcal{C}_{k,n}\frac{1}{(x_0^2+1)^{k+(n+1)/2}},$$

where $\mathcal{C}_{k,n} := \frac{(-1)^k}{\sqrt{\pi}}\frac{\Gamma(k+\frac{n+1}{2})}{\Gamma(k+\frac{n}{2})}$.

The inverse Fueter mapping theorem, II

Definition

Let n be an odd number. Let $\mathcal{P}_k(\underline{x})$ be an inner left spherical monogenic polynomial of degree $k \in \mathbb{N}_0$. We will denote by $\mathcal{W}^+_{k,n}(x)\mathcal{P}_k(\underline{x})$ and $\mathcal{W}^-_{k,n}(x)\mathcal{P}_k(\underline{x})$ the Fueter primitives of $\mathcal{F}^+_{k,n}(x)$ and $\mathcal{F}^-_{k,n}(x)$, that is $\mathcal{W}^+_{k,n}(x)\mathcal{P}_k(\underline{x})$ and $\mathcal{W}^-_{k,n}(x)\mathcal{P}_k(\underline{x})$ satisfy

$$\Delta^{k+\frac{n-1}{2}}(\mathcal{W}^+_{k,n}(x)\mathcal{P}_k(\underline{x})) = \mathcal{F}^+_{k,n}(x), \quad \Delta^{k+\frac{n-1}{2}}(\mathcal{W}^-_{k,n}(x)\mathcal{P}_k(\underline{x})) = \mathcal{F}^-_{k,n}(x).$$

The inverse Fueter mapping theorem, II

$$\mathcal{W}^{+}_{k,n}(x_0) = \frac{\mathcal{C}_{k,n}}{\mathcal{H}_{k,n}}\ D^{-(2k+n-1)} \frac{x_0}{(x_0^2+1)^{k+(n+1)/2}}$$

$$\mathcal{W}^{-}_{k,n}(x_0) := -\frac{\mathcal{C}_{k,n}}{\mathcal{H}_{k,n}}\ D^{-(2k+n-1)} \frac{1}{(x_0^2+1)^{k+(n+1)/2}}.$$

Replacing now x_0 by x in $\mathcal{W}^{\pm}_{k,n}(x_0)$ we get $\mathcal{W}^{\pm}_{k,n}(x)$ which are the required functions.

The inverse Fueter mapping theorem, II

Theorem (The inverse Fueter mapping theorem)

Let n be an odd number and let $\mathcal{P}_k(\underline{x})$ be an inner left spherical monogenic polynomial of degree $k \in \mathbb{N}_0$. Let

$$\tilde{f}(x)\mathcal{P}_k(\underline{x}) = (A(x_0, r) + \underline{\omega} B(x_0, r))\mathcal{P}_k(\underline{x})$$

be an axially monogenic function of degree k defined on an axially symmetric $U \subseteq \mathbb{R}^{n+1}$. Let Γ be the boundary of an open bounded subset $\mathcal{V}$ of the half plane $\mathbb{R} + \underline{\omega}\mathbb{R}^+$ and let $V \subset U$ be the open set in $\mathbb{R}^{n+1}$ induced by $\mathcal{V}$. Moreover suppose that Γ is a regular curve and consider the manifold

$$\Sigma := \{y_0 + \underline{\omega} r \,|\, (y_0, r) \in \Gamma, \quad \underline{\omega} \in \mathbb{S}^{n-1}\}.$$

The inverse Fueter mapping theorem, II

Then the function

$$f(x)\mathcal{P}_k(\underline{x})$$

$$= \int_\Gamma \mathcal{W}^-_{k,n}\Big(\frac{x-y_0}{r}\Big)\mathcal{P}_k\Big(\frac{\underline{x}-y_0}{r}\Big)r^{2k+n-2}[dy_0\, A(y_0,r) - dr\, B(y_0,r)]$$

$$-\int_\Gamma \mathcal{W}^+_{k,n}\Big(\frac{x-y_0}{r}\Big)\mathcal{P}_k\Big(\frac{\underline{x}-y_0}{r}\Big)r^{2k+n-2}[dy_0 B(y_0,r) + drA(y_0,r)].$$

is a Fueter's primitive of $\tilde{f}(x)\mathcal{P}_k(\underline{x})$ on V.

The inverse Fueter mapping theorem, II

Let us denote by $\mathcal{AM}_k(U)$ the set of axially monogenic functions of degree k on the axially symmetric open set U and let us introduce the set

$$\mathscr{N}_k(U) = \{\varphi_k = \sum_{j=1}^{m_k} f_j(x)\mathcal{P}_{k,j}(\underline{x}) \ \mid f_j \in \mathcal{N}(U)\}.$$

Corollary

Let n be an odd number and let U be an axially symmetric open set in $\mathbb{R}^{n+1}$. There is a map of $\mathbb{R}_n$-modules

$$\mathcal{AM}_k(U) \to \mathcal{N}_k(U),$$

such that $(A_k + \underline{\omega} B_k)\mathcal{P}_k = \Delta^{k+\frac{n-1}{2}}((\alpha_k + \underline{\omega}\beta_k)\mathcal{P}_k)$. *Moreover, there is a map*

$$\mathcal{M}(U) \to \bigoplus_k \Delta^k \mathcal{N}_k(U),$$

such that, given $\tilde{f} = \sum_k \tilde{f}_k \in \mathcal{M}(U)$, $f_k \in \mathcal{AM}_k(U)$, *there are* $\varphi_k \in \mathcal{N}_k$ *such that*

$$\tilde{f} = \Delta^{\frac{n-1}{2}} \sum_k \Delta^k \varphi_k.$$

The case of biaxially monogenic functions, III

Definition

Let U be an open set in $\mathbb{R}^p \times \mathbb{R}^q$ be invariant under the action of the group $\mathrm{Spin}(p) \times \mathrm{Spin}(q)$. Roughly speaking a monogenic function on U that is of the form

$$f(\underline{x},\underline{y}) = A(|\underline{x}|,|\underline{y}|) + \frac{\underline{x}}{|\underline{x}|}B(|\underline{x}|,|\underline{y}|) + \frac{\underline{y}}{|\underline{y}|}C(|\underline{x}|,|\underline{y}|) + \frac{\underline{x}}{|\underline{x}|}\frac{\underline{y}}{|\underline{y}|}D(|\underline{x}|,|\underline{y}|)$$

is said biaxially monogenic function on U. Both

$$\frac{\underline{x}}{|\underline{x}|}B(|\underline{x}|,|\underline{y}|) + \frac{\underline{y}}{|\underline{y}|}C(|\underline{x}|,|\underline{y}|), \tag{7}$$

and

$$A(|\underline{x}|,|\underline{y}|) + \frac{\underline{x}}{|\underline{x}|}\frac{\underline{y}}{|\underline{y}|}D(|\underline{x}|,|\underline{y}|), \tag{8}$$

are biaxially monogenic functions

The case of biaxially monogenic functions, III

T. Qian, F. Sommen, *Deriving harmonic functions in higher dimensional spaces*, Zeit. Anal. Anwen., **2** (2003), 1–12. shows that the Fueter mapping theorem can be extended to this setting.
Indeed, functions W of type

$$W(\underline{x},\underline{y}) = h_1(|\underline{x}|,|\underline{y}|)\frac{\underline{x}}{|\underline{x}|} + h_2(|\underline{x}|,|\underline{y}|)\frac{\underline{y}}{|\underline{y}|} \tag{9}$$

with h_1, h_2 real valued and such that W is in the kernel of the operator $\frac{\underline{x}}{|\underline{x}|}\partial_{|\underline{x}|} + \frac{\underline{y}}{|\underline{y}|}\partial_{|\underline{y}|}$, are such that

$$\Delta^{\frac{p+q}{2}-1} W(\underline{x},\underline{y}) = f(\underline{x},\underline{y}) \tag{10}$$

with f biaxially monogenic.

The case of biaxially monogenic functions, III

Definition

Let $\mathcal{U} \subseteq (\mathbb{R}^+ \cup \{0\}) \times (\mathbb{R}^+ \cup \{0\})$ and let $U \subseteq \mathbb{R}^p \times \mathbb{R}^q$ be the set induced by $\mathcal{U}$. Then we denote by $\mathcal{H}_B(U)$ the set of functions W of the form

$$W(\underline{x}, \underline{y}) = h_1(|\underline{x}|, |\underline{y}|)\frac{\underline{x}}{|\underline{x}|} + h_2(|\underline{x}|, |\underline{y}|)\frac{\underline{y}}{|\underline{y}|}$$

with h_1, h_2 real valued and such that W is in the kernel of the operator

$$\frac{\underline{x}}{|\underline{x}|}\partial_{|\underline{x}|} + \frac{\underline{y}}{|\underline{y}|}\partial_{|\underline{y}|}.$$

Lemma

Let U be an open set in $\mathbb{R}^p \times \mathbb{R}^q$, for $p \geq 1$ and $q \geq 1$, invariant under the action of the group $\mathrm{Spin}(p) \times \mathrm{Spin}(q)$ and let $\underline{x}/r \in \mathbb{S}^{p-1}$, $\underline{y}/\rho \in \mathbb{S}^{q-1}$, where $r = |\underline{x}|$, $\rho = |\underline{y}|$. Then the function W is in the kernel of the operator $\frac{\underline{x}}{|\underline{x}|}\partial_{|\underline{x}|} + \frac{\underline{y}}{|\underline{y}|}\partial_{|\underline{y}|}$ if and only if its components h_1 and h_2 satisfy the equations

$$\begin{cases} \partial_r h_1(r,\rho) + \partial_\rho h_2(r,\rho) = 0, \\ \partial_\rho h_1(r,\rho) - \partial_r h_2(r,\rho) = 0. \end{cases} \tag{11}$$

The case of biaxially monogenic functions, III

Proposition

Let U in $\mathbb{R}^p \times \mathbb{R}^q$, for $p \geq 1$ and $q \geq 1$, be invariant under the action of the group $\mathrm{Spin}(p) \times \mathrm{Spin}(q)$, and assume that $W \in \mathcal{H}_B(U)$. Then we have

$$W(\underline{x}, \underline{y}) = \mathrm{Re}\Big((h_1(r,\rho) + ih_2(r,\rho))(\underline{\omega} - i\underline{\nu})\Big). \quad (12)$$

Moreover, if we set

$$H(r - i\rho) := h_1(r,\rho) + ih_2(r,\rho), \quad H^{(\ell)}(r) := \partial_r^\ell H(r), \quad \ell = 0, 1, 2, ... \quad (13)$$

then W can be represented in power series as follows:

$$W(\underline{x}, \underline{y}) = \sum_{\ell=0}^{+\infty} \Big(\frac{1}{(2\ell)!} \underline{y}^{2\ell} H^{(2\ell)}(r) \frac{\underline{x}}{r} - \frac{1}{(2\ell+1)!} \underline{y}^{2\ell+1} H^{(2\ell+1)}(r) \Big). \quad (14)$$

The case of biaxially monogenic functions, III

Problem

Let f be a biaxially monogenic function on an open set $U \subseteq \mathbb{R}^p \times \mathbb{R}^q$, invariant under the action of the group $\mathrm{Spin}(p) \times \mathrm{Spin}(q)$, determine a function $W \in \mathcal{H}_B(U)$ such that

$$f(\underline{x}, \underline{y}) = \Delta^{\frac{m}{2}-1}(W(\underline{x}, \underline{y})),$$

where $m = p + q$, and p and q odd positive integers.

Theorem

Let U be a domain in $\mathbb{R}^3 \times \mathbb{R}^3$ invariant under the action of the group $\mathrm{Spin}(3) \times \mathrm{Spin}(3)$ and let $W \in \mathcal{H}_B(U)$. Then we have

$$(\Delta_{\underline{x}} + \Delta_{\underline{y}})^2 W(\underline{x}, \underline{y})|_{\underline{y}=0} = -8\partial_r \Big(\frac{1}{r}\partial_r^2 H(r)\Big)\underline{\omega},$$

where $\underline{x}/r = \underline{\omega}$.

Definition (The kernels $\mathcal{N}^+_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ and $\mathcal{N}^-_{p,q,\lambda,\mu}(\underline{x},\underline{y})$)

Let $p,q\in\mathbb{N}$ and let $\mathcal{G}(\underline{x}+\underline{y}-\underline{X}-\underline{Y})$ be the monogenic Cauchy kernel with $\underline{x}\in\mathbb{R}^p$, $\underline{y}\in\mathbb{R}^q$, and assume $\underline{\omega}\in\mathbb{S}^{p-1}$, $\underline{\eta}\in\mathbb{S}^{q-1}$ and for $\lambda>0$ and $\mu>0$, we define the kernels

$$\mathcal{N}^+_{p,q,\lambda,\mu}(\underline{x},\underline{y})=\frac{1}{A_{p+q}}\int_{\mathbb{S}^{p-1}}\int_{\mathbb{S}^{q-1}}\mathcal{G}(\underline{x}+\underline{y}-\lambda\underline{\xi}-\mu\underline{\eta})\,dS(\underline{\xi})\,dS(\underline{\eta}),\quad (15)$$

$$\mathcal{N}^-_{p,q,\lambda,\mu}(\underline{x},\underline{y})=\frac{1}{A_{p+q}}\int_{\mathbb{S}^{p-1}}\int_{\mathbb{S}^{q-1}}\mathcal{G}(\underline{x}+\underline{y}-\lambda\underline{\xi}-\mu\underline{\eta})\,\underline{\xi}\,\underline{\eta}\,dS(\underline{\xi})dS(\underline{\eta}),\quad (16)$$

where $dS(\underline{\xi})$ and $dS(\underline{\eta})$ are the scalar element of surface area of $\mathbb{S}^{p-1}$ and of $\mathbb{S}^{q-1}$, respectively.

Theorem (The restrictions of the kernels $\mathcal{N}^+_{p,q}(\underline{x},\underline{y})$ and $\mathcal{N}^-_{p,q}(\underline{x},\underline{y})$ to $\underline{y}=0$)

Let p, q be odd numbers. Let $\mathcal{N}^+_{p,q}(\underline{x},\underline{y})$ and $\mathcal{N}^-_{p,q}(\underline{x},\underline{y})$ be the kernels defined in (15) and (16), respectively. Then their restrictions to $\underline{y}=0$ are given by

$$\mathcal{N}^+_{p,q,\lambda,\mu}(\underline{x},0)=\frac{A_q A_{p-1}}{A_{p+q}}[J_{2,\lambda,\mu}(r;p,q)-r\lambda J_{1,\lambda,\mu}(r;p,q)]\frac{\underline{x}}{r} \qquad (17)$$

$$\mathcal{N}^-_{p,q,\lambda,\mu}(\underline{x},0)=\frac{A_q}{A_{p+q}}J_{2,\lambda,\mu}(r,p,q)\mu\frac{\underline{x}}{r} \qquad (18)$$

where the functions $J_{j,\lambda,\mu}(r;p,q)$, $j=1,2$ are given by

$$J_{j,\lambda,\mu}(r;p,q):=\int_{-1}^{1}\frac{t^{j-1}(1-t^2)^{(p-3)/2}}{(r^2-2r\lambda t+\lambda^2+\mu^2)^{(p+q)/2}}\,dt, \quad j=1,2. \qquad (19)$$

Theorem (The restrictions of the kernels $\mathcal{N}^{+}_{3,3,\lambda,\mu}(\underline{x},\underline{y})$ and $\mathcal{N}^{-}_{3,3,\lambda,\mu}(\underline{x},\underline{y})$ to $\underline{y}=0$)

Let $\mathcal{N}^{+}_{3,3,\lambda,\mu}(\underline{x},\underline{y})$ and $\mathcal{N}^{-}_{3,3,\lambda,\mu}(\underline{x},\underline{y})$ be the kernels defined in (15) and (16), respectively. Then their restrictions to $\underline{y}=0$ are given by

$$\mathcal{N}^{+}_{3,3,\lambda,\mu}(\underline{x},0)=\frac{A_3A_2}{A_6}\,\frac{2\lambda r(2-(r^2+\lambda^2+\mu^2))}{[(r^2+\lambda^2+\mu^2)^2-4\lambda^2r^2]^2}\,\frac{\underline{x}}{r}, \tag{20}$$

$$\mathcal{N}^{-}_{3,3,\lambda,\mu}(\underline{x},0)=\frac{A_3}{A_6}\,\frac{4\lambda\mu r}{[(r^2+\lambda^2+\mu^2)^2-4\lambda^2r^2]^2}\,\frac{\underline{x}}{r}. \tag{21}$$

Definition

Let p and q be an odd numbers and let $\lambda > 0$ and $\mu > 0$. We say that $\mathcal{W}^{+}_{p,q,\lambda,\mu}(\underline{x}, \underline{y})$ and $\mathcal{W}^{-}_{p,q,\lambda,\mu}(\underline{x}, \underline{y})$ are Fueter's primitives of $\mathcal{N}^{+}_{p,q,\lambda,\mu}(\underline{x}, \underline{y})$ and $\mathcal{N}^{-}_{p,q,\lambda,\mu}(\underline{x}, \underline{y})$, respectively, if they satisfy

$$\Delta^{\frac{(p+q)}{2}-1}(\mathcal{W}^{+}_{p,q,\lambda,\mu}(\underline{x}, \underline{y})) = \mathcal{N}^{+}_{p,q,\lambda,\mu}(\underline{x}, \underline{y}),$$

$$\Delta^{\frac{(p+q)}{2}-1}(\mathcal{W}^{-}_{p,q,\lambda,\mu}(\underline{x}, \underline{y})) = \mathcal{N}^{-}_{p,q,\lambda,\mu}(\underline{x}, \underline{y}).$$

Definition

Let p and q be an odd numbers and let $\lambda > 0$ and $\mu > 0$. Let $\mathcal{W}^{+}_{p,q,\lambda,\mu}(\underline{x}, \underline{y})$ and $\mathcal{W}^{-}_{p,q,\lambda,\mu}(\underline{x}, \underline{y})$ be Fueter primitives of $\mathcal{N}^{+}_{p,q,\lambda,\mu}(\underline{x}, \underline{y})$ and $\mathcal{N}^{-}_{p,q,\lambda,\mu}(\underline{x}, \underline{y})$. We denote by $H^{(\ell)}_{p,q,\lambda,\mu,\pm}(r)$, for $\ell \in \mathbb{N} \cup \{0\}$, the coefficients that appear in the series expansions:

$$\mathcal{W}^{\pm}_{p,q,\lambda,\mu}(\underline{x}, \underline{y})$$

$$= \sum_{\ell=0}^{+\infty} \Big(\frac{1}{(2\ell)!} \underline{y}^{2\ell} H^{(2\ell)}_{p,q,\lambda,\mu,\pm}(r) \frac{\underline{x}}{r} - \frac{1}{(2\ell+1)!} \underline{y}^{2\ell+1} H^{(2\ell+1)}_{p,q,\lambda,\mu,\pm}(r) \Big).$$

Theorem (The differential equations for coefficients of the restrictions of $\mathcal{W}^{\pm}_{3,3,\lambda,\mu}$)

The coefficients $H^{(\ell)}_{3,3,\lambda,\mu,\pm}(r)$, for $\ell \in \mathbb{N} \cup \{0\}$, in the series expansions:

$$\mathcal{W}^{\pm}_{3,3,\lambda,\mu}(\underline{x},\underline{y})$$

$$= \sum_{\ell=0}^{+\infty} \left(\frac{1}{(2\ell)!} \underline{y}^{2\ell} H^{(2\ell)}_{3,3,\lambda,\mu,\pm}(r) \frac{\underline{x}}{r} - \frac{1}{(2\ell+1)!} \underline{y}^{2\ell+1} H^{(2\ell+1)}_{3,3,\lambda,\mu,\pm}(r) \right)$$

of Fueter's primitives of $\mathcal{N}^{\pm}_{3,3,\lambda,\mu}(\underline{x},\underline{y})$ are given by the differential equations

$$-8\partial_r \left(\frac{1}{r} \partial_r^2 H_{3,3,\lambda,\mu,+}(r) \right) = \frac{A_3 A_2}{A_6} \; \frac{2\lambda r(2-(r^2+\lambda^2+\mu^2))}{[(r^2+\lambda^2+\mu^2)^2 - 4\lambda^2 r^2]^2},$$

$$-8\partial_r \left(\frac{1}{r} \partial_r^2 H_{3,3,\lambda,\mu,-}(r) \right) = \frac{A_3}{A_6} \; \frac{4\lambda\mu r}{[(r^2+\lambda^2+\mu^2)^2 - 4\lambda^2 r^2]^2}.$$

Theorem

Let $f(x)$ be a biaxially monogenic function of the form (where $\underline{\omega} = \underline{x}/r$, $r = |\underline{x}|$, $\underline{\nu} = \underline{y}/\rho$, $\rho = |\underline{y}|$), $f(\underline{x}, \underline{y}) = \underline{\omega} B(\rho, r) + \underline{\nu} C(\rho, r)$ defined on an axially symmetric domain $U \subseteq \mathbb{R}^{p+q}$, where p and q are odd numbers. Let Γ be the boundary of an open bounded subset $\mathcal{V}$ of the half plane $\underline{\xi}\mathbb{R}^+ + \underline{\eta}\mathbb{R}^+$ and let $V = \{\underline{\xi} u + \underline{\eta} v,\ (u, v) \in \mathcal{V},\ \underline{\xi} \in \mathbb{S}^{p-1},\ \ \underline{\eta} \in \mathbb{S}^{q-1}\} \subset U$. Moreover suppose, that Γ is a regular curve whose parametric equations $\lambda = \lambda(s)$, $\mu = \mu(s)$ are expressed in terms of the arc-length $s \in [0, L]$, $L > 0$. Then the function

$$W(\underline{x}, \underline{y}) := \int_\Gamma \mathcal{W}^+_{p,q,\lambda,\mu}(\underline{x}, \underline{y}) \mu^{p-1} \lambda^{q-1} \left[C(\lambda, \mu) d\lambda - B(\lambda, \mu) d\mu\right] \quad (22)$$

$$+ \int_\Gamma \mathcal{W}^-_{p,q,\lambda,\mu}(\underline{x}, \underline{y})\, \mu^{p-1} \lambda^{q-1} \left[B(\lambda, \mu) d\lambda + C(\lambda, \mu) d\mu\right],$$

is a Fueter's primitive of $f(\underline{x}, \underline{y})$ on U.

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The Fueter-Sce mappling in integral form

Theorem (Representation Formula (or Structure Formula))

Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric (s-domain) and let f be an s-monogenic function on U. For any vector $x = u + I_x v \in U$ the following formulas hold:

$$f(x) = \frac{1}{2}\Big[1 - I_x I\Big] f(u + Iv) + \frac{1}{2}\Big[1 + I_x I\Big] f(u - Iv) \tag{23}$$

The Fueter-Sce mappling in integral form

By the Representation Formula $\mathcal{SM}(U)$ consists of those functions

$$f(x_0, |\underline{x}|) = u(x_0, |\underline{x}|) + Iv(x_0, |\underline{x}|), \quad I \in \mathbb{S}$$

where $u, v: \ U \subset \mathbb{R}^{n+1} \to \mathbb{R}_n$,

The Fueter-Sce mappling in integral form

Describe the map $f \mapsto \breve{f} = \Delta_{n+1}^{\frac{n-1}{2}} f$, $f \in \mathcal{SM}(U)$ in integral form.

Provide a functional calculus based on axially monogenic functions.

The Fueter-Sce mappling in integral form

Cauchy formula with s-monogenic kernel

Let $U \subset \mathbb{R}^{n+1}$ be a bounded axially symmetric s-domain such that $\partial(U \cap \mathbb{C}_I)$ is union of a finite number of rectifiable Jordan curves for every $I \in \mathbb{S}$. Let f be a left s-monogenic function on an open set that contains U, $x \in U$ and set $ds_I = ds/I$, $ds = du + Idv$. Then

$$f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S^{-1}(s,x) ds_I f(s) \tag{24}$$

where

$$S^{-1}(s,x) = -(x^2 - 2x\mathrm{Re}(s) + |s|^2)^{-1}(x - \bar{s})$$

and the integral does not depend on U and on the imaginary unit $I \in \mathbb{S}$.

The Fueter-Sce mappling in integral form

Theorem:
Let $x,\ s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. Then the following identity holds:

$$-(x^2 - 2x\mathrm{Re}(s) + |s|^2)^{-1}(x - \bar{s}) = (s - \bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-1}.$$

Definition

Let $x,\ s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$.

- We say that $S^{-1}(s,x)$ is written in the form I if

$$S^{-1}(s,x) := -(x^2 - 2x\mathrm{Re}(s) + |s|^2)^{-1}(x - \bar{s}).$$

- We say that $S^{-1}(s,x)$ is written in the form II if

$$S^{-1}(s,x) := (s - \bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-1}.$$

The Fueter-Sce mappling in integral form

Remark

Even though $S^{-1}(s,x)$ written in the form I is more suitable for several applications, for example for the definition of a functional calculus, it does not allow easy computation of the powers of the Laplacian

$$\Delta = \Delta_{n+1} = \partial_{x_0}^2 + \partial_{x_1}^2 + \ldots + \partial_{x_n}^2$$

applied to it. The form II is the one that allows, by iteration, the computation of $\Delta^{\frac{n-1}{2}} S^{-1}(s,x)$.

The Fueter mapping theorem in integral form

Theorem (Explicit computation of $\Delta^{\frac{n-1}{2}} S^{-1}(s,x)$)

Let $x, s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. Let $S^{-1}(s,x) = (s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-1}$ be the slice-monogenic Cauchy kernel and let $\Delta = \sum_{i=0}^{n} \frac{\partial^2}{\partial x_i^2}$ be the Laplace operator in the variable x. Then, for $h \geq 1$, we have:

$$\Delta^h S^{-1}(s,x) = C_{n,h}(s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-(h+1)}.$$

where

$$C_{n,h} := (-1)^h \prod_{\ell=1}^{h}(2\ell) \prod_{\ell=1}^{h}(n-(2\ell-1)).$$

The Fueter mapping theorem in integral form

Theorem

Let x, $s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. Then the function

$$\Delta^h S^{-1}(s,x)$$

is a right s-monogenic function in the variable s, for any $h \in \mathbb{N}$.

Theorem

Let n be an odd number and let x, $s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. Then the function

$$\Delta^{\frac{n-1}{2}} S^{-1}(s,x)$$

is a monogenic function in the variable x.

The Fueter mapping theorem in integral form

Definition (The $\mathcal{F}_n$-kernel)

Let n be an odd number. Let x, $s \in \mathbb{R}^{n+1}$. We define, for $s \notin [x]$, the $\mathcal{F}_n$-kernel as

$$\mathcal{F}_n(s,x) := \Delta^{\frac{n-1}{2}} S^{-1}(s,x) = \gamma_n (s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}},$$

where

$$\gamma_n := (-1)^{(n-1)/2} 2^{n-1} \left(\left(\frac{n-1}{2} \right)! \right)^2.$$

The Fueter mapping theorem in integral form

Theorem (The Fueter mapping theorem in integral form)

Let n be an odd number. Let $U \subset \mathbb{R}^{n+1}$ be a bounded axially symmetric s-domain such that $\partial(U \cap \mathbb{C}_I)$ is union of a finite number of rectifiable Jordan curves for every $I \in \mathbb{S}$. Let f be a left s-monogenic function on an open set that contains U, $x \in U$ and set $ds_I = ds/I$, $ds = du + Idv$. Then, if $x \in U$, the function $\breve{f}(x)$ given by $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$ is monogenic and it admits the integral representation

$$\breve{f}(x) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} \mathcal{F}_n(s,x) ds_I f(s), \quad ds_I = ds/I, \tag{25}$$

where the integral does not depend on U nor on the imaginary unit $I \in \mathbb{S}$.

The F-functional calculus for bounded operators

- V_n is two-sided Banach module $V \otimes \mathbb{R}_n$ over $\mathbb{R}_n$.
- An element in V_n is of the type $\sum_A v_A \otimes e_A$ is a multi-index.
- Finally, we define $\|v\|_{V_n} = \sum_A \|v_A\|_V$.
- $\mathcal{B}(V)$ the space of bounded $\mathbb{R}$-homomorphisms of the Banach space V to itself endowed with the natural norm denoted by $\|\cdot\|_{\mathcal{B}(V)}$.
- Given $T_A \in \mathcal{B}(V)$, we can introduce the operator $T = \sum_A T_A e_A$ and its action on $v = \sum v_B e_B \in V_n$ as $T(v) = \sum_{A,B} T_A(v_B) e_A e_B$.

The F-functional calculus for bounded operators

- The operator T is a module homomorphism which is a bounded linear map on V_n.
- The set of all such bounded operators, with commuting components, is denoted by $\mathcal{BC}_n(V_n)$.
- We define $\|T\|_{\mathcal{BC}_n(V_n)} = \sum_A \|T_A\|_{\mathcal{BC}(V)}$.
- $\mathcal{BC}_n^{0,1}(V_n)$ is the space of operators of the form $T = T_0 + \sum_{j=1}^n T_j e_j$ where $T_\mu \in \mathcal{B}(V)$ for $\mu = 0, 1, ..., n$ such that T_μ commute among themselves.

The F-functional calculus for bounded operators

Definition

Let n be an odd number, and let $m \in \mathbb{N}$, then we set

$$\mathcal{P}_{m,n}(x) := \Delta^{\frac{n-1}{2}} x^m. \tag{26}$$

In the sequel we omit the index n in $\mathcal{P}_{m,n}(x)$ and we simply write $\mathcal{P}_m(x)$.

The F-functional calculus for bounded operators

Lemma

Let n be an odd number, and $s, x \in \mathbb{R}^{n+1}$. Then the series

$$\sum_{m \geq n-1} \mathcal{P}_m(x) s^{-1-m},$$

converges if and only if $|x| < |s|$.

The F-functional calculus for bounded operators

Definition (Monogenic Cauchy kernel series)

Let $s, x \in \mathbb{R}^{n+1}$ where n is an odd number. We define the monogenic Cauchy kernel series as

$$\mathcal{F}_{\Sigma}(s,x) := \sum_{m \geq n-1} \mathcal{P}_m(x) s^{-1-m},$$

for $|x| < |s|$.

The F-functional calculus for bounded operators

Definition (Monogenic Cauchy kernel operator series)

Let n be an odd number, and let $T \in \mathcal{BC}^{0,1}_n(V_n)$ with $\|T\| < |s|$, where $s \in \mathbb{R}^{n+1}$. We define the monogenic Cauchy kernel operator series as:

$$\mathcal{F}_\Sigma(s, T) := \sum_{m \geq n-1} \mathcal{P}_m(T) s^{-1-m},$$

where we have substituted the operator T_i for x_i in the polynomials $\mathcal{P}_m(x)$.

The F-functional calculus for bounded operators

Theorem

Let n be an odd number, $T \in \mathcal{BC}_n^{0,1}(V_n)$ with $\|T\| < |s|$, where $s \in \mathbb{R}^{n+1}$. Then we have

$$\sum_{m \geq n-1} \mathcal{P}_m(T) s^{-1-m} = \gamma_n (s\mathcal{I} - \overline{T})(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-\frac{n+1}{2}},$$

where

$$\gamma_n := (-1)^{(n-1)/2} 2^{n-1} \Big((\frac{n-1}{2})! \Big)^2.$$

The F-functional calculus for bounded operators

Definition (The $\mathcal{F}$-spectrum and the $\mathcal{F}$-resolvent sets)

Let n be an odd number and let $T \in \mathcal{BC}_n^{0,1}(V_n)$. We define the $\mathcal{F}$-spectrum $\sigma_{\mathcal{F}}(T)$ of T as:

$$\sigma_{\mathcal{F}}(T) = \{s \in \mathbb{R}^{n+1} \quad : \qquad s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T} \quad \textit{is not invertible}\}.$$

The $\mathcal{F}$-resolvent set $\rho_{\mathcal{F}}(T)$ is defined by

$$\rho_{\mathcal{F}}(T) = \mathbb{R}^{n+1} \setminus \sigma_{\mathcal{F}}(T).$$

The F-functional calculus for bounded operators

Theorem (Compactness of the $\mathcal{F}$-spectrum)

Let n be an odd number, $T \in \mathcal{B}_n^{0,1}(V_n)$ with commuting components. Then the $\mathcal{F}$-spectrum $\sigma_{\mathcal{F}}(T)$ is a compact nonempty set. Moreover $\sigma_{\mathcal{F}}(T)$ is contained in $\{s \in \mathbb{R}^{n+1} \ : \ |s| \leq \|T\| \ \}$.

The F-functional calculus for bounded operators

Theorem (Structure of the $\mathcal{F}$-spectrum)

Let $T \in \mathcal{BC}_n^{0,1}(V_n)$ and let $p = p_0 + p_1 I \in [p_0 + p_1 I] \subset \mathbb{R}^{n+1} \setminus \mathbb{R}$, such that $p \in \sigma_{\mathcal{F}}(T)$. Then all the elements of the $(n-1)$-sphere $[p_0 + p_1 I]$ belong to $\sigma_{\mathcal{F}}(T)$. Thus the $\mathcal{F}$-spectrum consists of real points and/or $(n-1)$-spheres.

Definition ($\mathcal{F}$-resolvent operator)

Let n be an odd number, $s \in \mathbb{R}^{n+1}$ and let $T \in \mathcal{BC}_n^{0,1}(V_n)$. For $s \in \rho_{\mathcal{F}}(T)$ we define the $\mathcal{F}$-resolvent operator by

$$\mathcal{F}_n(s, T) := \gamma_n(s\mathcal{I} - \overline{T})(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-\frac{n+1}{2}}.$$

The F-functional calculus for bounded operators

Definition

- Let $T \in \mathcal{BC}^{0,1}_n(V_n)$ and let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric s-domain that contains the $\mathcal{F}$-spectrum $\sigma_{\mathcal{F}}(T)$, such that $\partial(U \cap \mathbb{C}_I)$ is union of a finite number of rectifiable Jordan curves for every $I \in \mathbb{S}$.
- Let W be an open set in $\mathbb{R}^{n+1}$. A function $f \in \mathcal{SM}(W)$ is said to be locally s-monogenic on $\sigma_{\mathcal{F}}(T)$ if there exists a domain $U \subset \mathbb{R}^{n+1}$, as above and such that $\overline{U} \subset W$, on which f is s-monogenic.
- We will denote by $\mathcal{SM}_{\sigma_{\mathcal{F}}(T)}$ the set of locally s-monogenic functions on $\sigma_{\mathcal{F}}(T)$.

The F-functional calculus for bounded operators

Theorem

Let n be an odd number, $T \in \mathcal{BC}_n^{0,1}(V_n)$*, let* $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(T)}$*. Let U be an open set, containing* $\sigma_{\mathcal{F}}(T)$*, as above. Then the integral*

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} \mathcal{F}_n(s,T)\, ds_I\, f(s), \qquad ds_I = ds/I \tag{27}$$

is independent of $I \in \mathbb{S}$ *and of the open set U.*

The F-functional calculus for bounded operators

Definition (The $\mathcal{F}$-functional calculus)

Let n be an odd number, $T \in \mathcal{BC}_n^{0,1}(V_n)$. Let U be an open set, containing $\sigma_{\mathcal{F}}(T)$, as above. Suppose that $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(T)}$ and let $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$. We define the $\mathcal{F}$-functional calculus as

$$\breve{f}(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} \mathcal{F}_n(s, T)\, ds_I\, f(s).$$

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Corollary

Let n be an odd number, $T \in \mathcal{BC}_n^{0,1}(V_n)$ and let $U \supset \sigma_{\mathcal{F}}(T)$ be as above. Then

$$\mathcal{P}_m(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} \mathcal{F}_n(s, T)\, ds_I\, s^m$$

and the integral does not depend on the open set U nor on $I \in \mathbb{S}$.

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Definition

Let n be an odd number. Let $\{T_m\}_{m\in\mathbb{N}}$ and T belong to $\mathcal{BC}^{0,1}(V_n)$, suppose that $\rho_F(T) = \rho_F(T_m)$ for all $m \in \mathbb{N}$. We say that T_m converges to T in the norm F-resolvent sense if $\mathcal{F}_n(s, T_m) \to \mathcal{F}_n(s, T)$ as $m \to \infty$ for all $s \in \rho_F(T)$.

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Theorem

Let n be an odd number. Let $\{T_m\}_{m\in\mathbb{N}}$ and T be elements in $\mathcal{BC}^{0,1}(V_n)$, suppose that $\rho_F(T)=\rho_F(T_m)$ for all $m\in\mathbb{N}$. Then $T_m\to T$ in the norm if and only if $T_m\to T$ in the norm F-resolvent sense.

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Theorem

Let n be an odd number. Let f and $g \in \mathcal{SM}_{\sigma_F(T)}$ and let $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$ and $\breve{g}(x) = \Delta^{\frac{n-1}{2}} g(x)$. Then we have

$$(\breve{f} + \breve{g})(T) = \breve{f}(T) + \breve{g}(T), \qquad (\breve{f}\lambda)(T) = \breve{f}(T)\lambda, \qquad \text{for all } \ \lambda \in \mathbb{R}_n.$$

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Theorem

Let n be an odd number and let $T \in \mathcal{BC}(V_n)$. Let $f(s) = \sum_{\ell \geq 0} s^\ell a_\ell$ where $a_\ell \in \mathbb{R}_n$ be such that $f \in \mathcal{SM}_{\sigma_F(T)}$ and let $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$. Then we have

$$\breve{f}(T) = \sum_{\ell \geq 0} \mathcal{P}_{\ell,n}(T) a_\ell.$$

where $\mathcal{P}_{\ell,n}(T)$ has been obtained by replacing x by T in the polynomials $\mathcal{P}_{\ell,n}(x) := \Delta^{\frac{n-1}{2}} x^\ell$.

The F-functional calculus for bounded operators

Theorem (The F-resolvent equation)

Let n be an odd number and let $T \in \mathcal{BC}(V_n)$. Let $s \in \rho_F(T)$ then $\mathcal{F}_n(s,T)$, satisfies the equation

$$\mathcal{F}_n(s,T)s - T\mathcal{F}_n(s,T) = \gamma_n \mathcal{Q}_s(T)^{\frac{n-1}{2}}, \tag{28}$$

where

$$\mathcal{Q}_s(T) := (s^2\mathcal{I} - s(T+\overline{T}) + T\overline{T})^{-1}.$$

The F-functional calculus for bounded operators

Theorem (Bounded perturbations)

Let n be an odd number, $T, Z \in \mathcal{BC}_n^{0,1}(V_n)$, $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(T)}$ and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that, for $\|Z - T\| < \delta$, we have $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(Z)}$ and

$$\|\breve{f}(Z) - \breve{f}(T)\| < \varepsilon.$$

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